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The strong nonlinear interaction of Tollmien–Schlichting waves and Taylor–Görtler vortices in curved channel flow

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Viscous fluid flows with curved streamlines can support both centrifugal and viscous travelling wave instabilities. Here the interaction of these instabilities in the context of the fully developed flow in a curved channel is discussed. The viscous (Tollmien–Schlichting) instability is described asymptotically at high Reynolds numbers and it is found that it can induce a Taylor–Görtler flow even at extremely small amplitudes. In this interaction, the Tollmien–Schlichting wave can drive a vortex state with wavelength either comparable with the channel width or the wavelength of lower-branch viscous modes. The nonlinear equations which describe these interactions are solved for nonlinear equilibrium states.

1. Introduction

There are many fluid flows of practical importance where destabilizing centrifugal and viscous instability mechanisms are both present. Thus, for example, the flow in a curved rectangular duct or the flow over parts of a laminar flow wing (see Harvey & Pride 1982) can support both Taylor–Görtler vortices and Tollmien–Schlichting waves. Indeed the latter flow can also support Rayleigh instabilities associated with the highly inflexional velocity profiles in some directions. In these flows the possibility exists that the nonlinear interaction of the different instability mechanisms might produce premature transition to turbulence. In that regard a potentially significant result from the recent work of Hall & Smith (1988) is that interacting oblique Tollmien–Schlichting waves can generate longitudinal vortices essentially identical to Taylor–Görtler vortices even in the absence of wall curvature.

One of the first calculations on the interaction of Tollmien–Schlichting (TS) waves and Görtler vortices was given by Nayfeh (1981). Nayfeh discussed the effect of a Görtler vortex of given size on the growth of oblique TS waves. The amplitude of the vortex was assigned arbitrarily however and an eigenfunction shape found by solving the parallel-flow-approximation Görtler-vortex equations. It is now known (Hall 1982*a*) that, in the wavenumber regime considered by Nayfeh, these equations have solutions of no relevance to spatially growing vortices. Furthermore, the amplitude of the vortex cannot be assigned arbitrarily; it must of course be determined by either a numerical or analytical solution of the Navier–Stokes equation as in Hall (1982*b*, 1988). Having made these approximations Nayfeh found that the vortices could have a massive effect on the growth of TS waves. Later Malik (1986) showed that Nayfeh's numerical calculations were incorrect and that his conclusions were in error.

More recently Bennett & Hall (1988) examined the effect of finite-amplitude

Görtler vortices in fully developed flows on the growth of lower-branch TS waves. Here the nonlinear vortex state was found by solving the Navier–Stokes equations and a linear stability analysis of the solutions was given. It was shown that even small-amplitude vortices can have a significant effect on TS growth rates. The vortex flow found by Bennett & Hall had a wavelength comparable with the depth of the channel in which the flow occurred. The asymptotic structure of TS waves appropriate to the lower branch of the neutral curve corresponds in general to TS waves with a small spanwise wavenumber and Bennett & Hall showed how this structure could be modifed to allow for a faster spanwise dependence induced by the vortices.

The first nonlinear description of the interaction of vortices and TS waves was given by Hall & Smith (1988). Here it was shown that long-wavelength vortices and oblique TS waves undergo a resonant triad interaction at small amplitude. The interaction is, in the first instance, governed by ordinary differential triad amplitude equations which possess a finite time singularity. At higher amplitude the interaction is controlled by a coupled partial differential-ordinary integro-differential equation system. The solution of the system was found to depend crucially on the orientation of the TS waves to the vortices. The most dangerous type of interaction concerns TS waves propagating in a direction making an angle of more than 41.6° to the vortices since the resulting interaction produces a finite time singularity.

The interaction problem formulated for channel flows by Hall & Smith (1988) can be modified to take account of a basic state which evolves in the flow direction. Thus Hall & Smith (1989*a*) developed a related theory to describe the interaction of longitudinal vortices with two small-amplitude TS waves propagating at equal angles to the flow direction. Again it was found that singular solutions are possible if the angle of propagation of the TS waves is chosen appropriately. Subsequently Hall & Smith (1989*b*) investigated the more strongly nonlinear state which is implied by a particular singular solution found by Hall & Smith (1989*a*). In that strongly nonlinear situation it is possible to describe the key features of one form of boundary-layer transition. Thus Hall & Smith (1989*b*) describe how a two-dimensional TS wave undergoes a secondary instability to a pair of oblique TS waves and induces a longitudinal vortex structure. In fact the strongly nonlinear flow set up after the longitudinal vortex structure has developed ultimately develops a singularity at a finite downstream location; the flow patterns associated with all the stages described by Hall & Smith (1989*b*) have many similarities with those observed experimentally during transition.

Here we develop a strongly nonlinear theory to describe one of the larger-amplitude states implied by the investigation of Hall & Smith (1988) for channel flows. The interaction we describe is related to that of Hall & Smith (1989*b*) since the TS waves we introduce into the flow are sufficiently large to generate a longitudinal vortex field with downstream velocity component comparable with the unperturbed flow. The size of the TS waves is fixed by the condition that the downstream velocity component is perturbed at zeroth order by the vortex induced by the interacting TS waves. Thus this interaction leads to an $O(1)$ change from the basic state that exists in the absence of vortices. If the size of the TS wave is decreased then a weakly nonlinear bifurcation governed by a cubic-order amplitude equation typical of those found when using the Stuart–Watson method is retrieved. Surprisingly, the stronger type of interaction can occur at extremely small TS amplitudes both with and without wall curvature being present. The fact that such a relatively small TS wave can have such a large effect on the basic state is due to the large initial forces associated with the small streamwise lengthscale of the waves. The nonlinear equilibrium states appropriate to this interaction must be found numerically. As a special limiting case we also consider the situation when the induced vortex flow has spanwise wavelength comparable with the depth of the channel rather than the wavelength of a lower-branch TS wave. We note here that other nonlinear states are accessible by routes other than that described in this paper; see, for example, Hall & Smith (1989*b*) where the vortex–TS interaction for external flows is discussed. The question of whether the present route described is physically the most relevant or whether one of the 'by-pass' routes dominates in an experiment cannot yet be answered.

In addition, at small vortex wavenumbers the three-dimensional breakdown of a two-dimensional TS wave can be described by an analysis of our interaction equations. Thus we determine the size of two-dimensional TS waves which are neutrally stable to oblique TS waves. A key feature of this secondary instability process is the longitudinal vortex system induced by the interacting oblique waves. Furthermore the interaction can occur in a straight channel, thus yielding a mechanism for the breakdown of two-dimensional TS waves in parallel flows.

We note that the procedure adopted in this paper and the related work of Hall & Smith (1988, 1989*a*, *b*) is based on the high-Reynolds-number limit. Whilst this is of course unavoidable for the boundary-layer case, the channel flow stability problem can be investigated at finite Reynolds numbers. Indeed Daudpota, Hall & Zang (1988) have investigated the TS–vortex interaction problem for channel flows at finite Reynolds numbers. However, at finite Reynolds numbers it is only possible to make any analytical progress with exceedingly small disturbances which are almost neutral on the basis of linear theory. Thus the work of Daudpota *et al.* is restricted to interactions which are too small to have an $O(1)$ effect on the unperturbed state. Hence there is no overlap between the work discussed here and that of Daudpota *et al.*, but in a limiting small-amplitude form our earlier calculation (Hall & Smith 1988) reduces to the situation considered by Daudpota *et al.* and is consistent with the conclusions of the latter authors. An extension of the approach of Daudpota *et al.* into anything other than a weakly nonlinear state can only be done by solving the full Navier–Stokes equations numerically; we feel that it is more instructive to try and make some analytical progress by taking the limit of high Reynolds number.

The procedure adopted in the rest of this paper is as follows: in §2 the nonlinear equations governing fully nonlinear vortex flows in curved channels are described; in §3 these equations are solved in the presence of vortices of $O(1)$ cross-stream wavenumber and the possible equilibrium states are described; in §4 the corresponding calculation for small-wavenumber vortices is described; and in §5 the instability of two-dimensional TS waves in a straight channel is discussed through the interaction equations derived in §4. Finally in §6 we discuss our results and draw some conclusions, mainly that nonlinear effects lead to a supercritical bifurcation to a mixed vortex–TS state.

2. Formulation of the disturbance equations for curved channel flows

Consider the flow of a viscous incompressible fluid in a curved channel with walls defined by $r^* = a, a+d$ with respect to cylindrical polar coordinates (r^*, θ^*, z^*). It is assumed that the curvature parameter, δ, defined by

$$\delta = \frac{d}{a}, \tag{2.1}$$

is small. The flow is driven by the streamwise pressure gradient

$$\frac{1}{\rho}\frac{\partial p^*}{\partial \theta^*} = \frac{-12\nu V_{\mathrm{m}} r^*}{d^2}, \tag{2.2}$$

where ρ and ν are the density and kinematic viscosity of the fluid respectively, whilst V_{m} is a typical streamwise flow speed. The pressure gradient (2.2) drives a velocity field v_{m} in the streamwise direction and

$$v_{\mathrm{m}} = V_{\mathrm{m}}\, U(y), \tag{2.3}$$

where for small values of δ

$$U(y) = U_0(y) + O(\delta), \tag{2.4}$$

with

$$U_0(y) = 6y(1-y), \quad y = \frac{r^*-a}{d}. \tag{2.5a, b}$$

The dimensionless variables x, z, and t are then defined by

$$x = \frac{a\theta^*}{d}, \quad z = \frac{z^*}{d}, \quad t = \frac{t^*\nu}{d^2}. \tag{2.6a, b, c}$$

where t^* denotes the (dimensional) time. If (u, v, w) is the velocity field scaled on V_{m} with respect to (x, y, z) and the pressure is scaled on ρV_{m}^2 then the Navier–Stokes equations can be written in the form

$$\frac{1}{\mathscr{F}}\frac{\partial u}{\partial x} + \frac{\delta u}{\mathscr{F}} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7a}$$

$$\frac{1}{Re}\left\{\nabla^2 - \partial_t - \frac{\delta^2}{\mathscr{F}^2}\right\}u + \frac{2\delta}{Re\,\mathscr{F}^2}\frac{\partial v}{\partial x} - \frac{1}{\mathscr{F}}\frac{\partial p}{\partial x} = \mathrm{N}u + \frac{\delta uv}{\mathscr{F}}, \tag{2.7b}$$

$$\frac{1}{Re}\left\{\nabla^2 - \partial_t - \frac{\delta^2}{\mathscr{F}^2}\right\}v - \frac{2\delta}{Re\,\mathscr{F}^2}\frac{\partial u}{\partial x} - \frac{\partial p}{\partial y} = \mathrm{N}v - \frac{\delta u^2}{\mathscr{F}}, \tag{2.7c}$$

$$\frac{1}{Re}\{\nabla^2 - \partial_t\}\, w - \frac{\partial p}{\partial z} = \mathrm{N}w, \tag{2.7d}$$

where the Reynolds number Re is defined by

$$Re = V_{\mathrm{m}}\frac{d}{\nu} \tag{2.8a}$$

and

$$\mathscr{F} = 1 + \delta y, \quad \nabla^2 \equiv \frac{1}{\mathscr{F}^2}\partial_x^2 + \partial_y^2 + \frac{\delta}{\mathscr{F}}\partial_y + \partial_z^2, \tag{2.8b, c}$$

$$\mathrm{N} = (u/\mathscr{F})\,\partial_x + v\,\partial_y + w\,\partial_z. \tag{2.8d}$$

The Taylor number T is then defined by

$$T = 4Re^2\delta, \tag{2.9}$$

and it is known from the work of Dean (1928) that instability in the form of Taylor vortices occurs first for $O(1)$ spanwise wavelengths with T also an $O(1)$ quantity. In view of the smallness of δ it follows from (2.9) that Taylor vortices first occur at high values of the Reynolds number. The flow is therefore susceptible to lower-branch Tollmien–Schlichting instabilities with wavenumbers of order $Re^{-\frac{1}{7}}$ in the azimuthal direction. It is the nonlinear interaction of these two modes of instability which will be discussed in the next section.

However, because a primary aim of the present calculation is to shed light on the related external flow problem where long-wavelength Görtler vortices are important in the initial development of the flow, we consider a closely related interaction problem arising from the weakly nonlinear theory of Hall & Smith (1988). Here the Taylor vortices occur with spanwise wavenumbers $O(Re^{-\frac{1}{7}})$ which are comparable with the wavenumbers of lower-branch Tollmien–Schlichting waves. These long-wavelength vortices occur at relatively high values of the Taylor number $T \sim O(Re^{\frac{2}{7}})$ which requires that $Re \sim O(\delta^{-\frac{7}{12}})$ whereas the previous interaction has $Re \sim O(\delta^{-\frac{1}{2}})$. Thus, for a given value of Re, the former type of interaction occurs first when δ is increased from zero. Alternatively, for a given channel with δ fixed, the former interaction occurs at the lowest values of Re. However, there is some indication from the work of Hall & Smith (1988) that the second interaction can occur even at zero values of T, and so it is important to understand its structure.

Finally in this section we write down the form of the solutions of (2.7) appropriate to fully nonlinear Taylor vortices. Firstly, for $O(1)$ vortex wavelengths the pressure p and velocity field (u, v, w) can be written as

$$p = \left[-\frac{12x}{Re} + \frac{\tilde{p}}{2Re^2}\right][1 + O(Re^{-1})], \tag{2.10a}$$

$$(u, v, w) = \left[U_0 + \tilde{u}, \frac{\tilde{v}}{2Re}, \frac{\tilde{w}}{2Re}\right][1 + O(Re^{-1})]. \tag{2.10b}$$

Here $\tilde{u}, \tilde{v}$ and $\tilde{w}, \tilde{p}$ are functions of t, y and z and satisfy

$$\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{2.11a}$$

$$[\nabla_2^2 - \partial_t]\,\tilde{u} - \tfrac{1}{2}\tilde{v}U_0' = \tfrac{1}{2}(\tilde{v}\tilde{u}_y + \tilde{w}\tilde{u}_z), \tag{2.11b}$$

$$[\nabla_2^2 - \partial_t]\,\tilde{v} - \frac{\partial \tilde{p}}{\partial y} + \tfrac{1}{2}T\tilde{u}U_0 = \tfrac{1}{2}(\tilde{v}\tilde{v}_y + \tilde{w}\tilde{v}_z), \tag{2.11c}$$

$$[\nabla_2^2 - \partial_t]\,\tilde{w} - \frac{\partial \tilde{p}}{\partial z} = \tfrac{1}{2}(\tilde{v}\tilde{w}_y + \tilde{w}\tilde{w}_z) \tag{2.11d}$$

$$\tilde{u} = \tilde{v} = \tilde{w} = 0, \quad y = 0, 1, \tag{2.11e}$$

with $\nabla_2^2 = \partial_y^2 + \partial_z^2$.

The solution of the linearized form of the above equations shows that Taylor vortices grow exponentially in time for $T > T_c \sim 5162$, and weakly nonlinear theory (Seminara 1975) can then be used to show that the most unstable mode is stabilized by nonlinear effects. The neutral curve for the linear stability problem is given by, for example, Hall (1982*b*).

In the fully nonlinear regime possible stable finite-amplitude solutions of (2.11) can be found by stepping the equations forward in time until the flow equilibrates. Bennett & Hall (1988) investigated the instability of these vortex flows to small Tollmien–Schlichting waves. Here in contrast we allow the Tollmien–Schlichting waves to be sufficiently large that they have an $O(1)$ effect on the vortex flows.

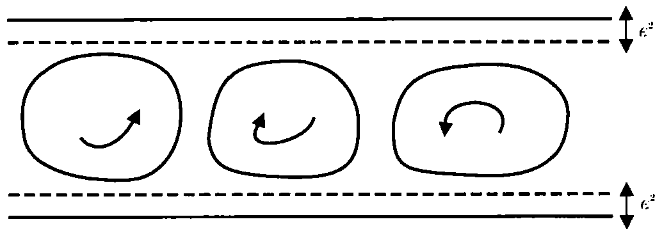

FIGURE 1. The flow structure and associated scales for the $O(1)$-wavelength vortex interaction problem.

Figure 1 shows a summary of the flow structure and the associated scales for the $O(1)$-wavelength interaction problem. In the following section we discuss the situation when the vortex wavelength lengthens and becomes comparable with the wavelength of a lower-branch TS wave.

Second, if the vortex wavelength is $O(Re^{-\frac{1}{7}})$ we define

$$Z = Re^{-\frac{1}{7}}z, \quad \delta = \tfrac{1}{4}\tilde{T}Re^{-\frac{12}{7}}, \tag{2.12a, b}$$

and (2.10) is modified to give

$$p = \left[\frac{-12x}{Re} + \frac{\tilde{p}}{2Re^{\frac{12}{7}}}\right][1 + O(Re^{-\frac{5}{7}})], \tag{2.13a}$$

$$[u, v, w] = \left[U_0 + \tilde{u}, \frac{\tilde{v}}{2Re}, \frac{\tilde{w}}{2Re^{\frac{6}{7}}}\right][1 + O(Re^{-\frac{5}{7}})]. \tag{2.13b}$$

The functions $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$ are now dependent on y, Z, and t and satisfy

$$\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial Z} = 0, \tag{2.14a}$$

$$\{\partial_y^2 - \partial_t\}\,\tilde{u} = \tfrac{1}{2}\tilde{v}U_0' + \tfrac{1}{2}(\tilde{v}\tilde{u}_y + \tilde{w}\tilde{u}_Z), \tag{2.14b}$$

$$\tfrac{1}{2}\tilde{T}\tilde{u}U_0 - \frac{\partial \tilde{p}}{\partial y} = 0, \tag{2.14c}$$

$$\{\partial_y^2 - \partial_t\}\,\tilde{w} - \frac{\partial \tilde{p}}{\partial Z} = \tfrac{1}{2}\{\tilde{v}\tilde{w}_y + \tilde{w}\tilde{w}_Z\}, \tag{2.14d}$$

$$\tilde{u} = \tilde{v} = \tilde{w} = 0, \quad y = 0, 1. \tag{2.14e}$$

The linearized form of this system has unstable disturbances for any non-zero value of $\tilde{T}$. Indeed the neutral value of $\tilde{T}$ is proportional to k^{-2} if k is the spanwise wavenumber. This means that finite-amplitude solutions of (2.14) cannot be found by integrating forward in time since the energy of the disturbance will cascade into the higher harmonics. We return to the long-wavelength limit in §4, since we concentrate next on the nonlinear interaction problem for $\partial/\partial z = O(1)$.

3. The nonlinear evolution equations for TS waves and Taylor vortices of $O(1)$ wavelength

Since the only major difference between the present calculation and that of Bennett & Hall (1988) is that the TS wave now has an $O(1)$ effect on the vortex, only the essential details of the flow structure will be given. The size of the TS wave is fixed by the requirement that the nonlinear terms in the y- and z-momentum equations driven by the TS waves be comparable with those in (2.7*c*, *d*). This forcing is most important away from the viscous wall layers of thickness $Re^{-\frac{2}{7}}$ in which the TS waves adjust to the no-slip condition at the wall. It is convenient at this stage to define the small parameter ϵ by

$$\epsilon = Re^{-\frac{1}{7}}.$$

Suppose then that the TS wave is proportional to

$$E = \exp\left[\mathrm{i}\left(\alpha\epsilon x - \int^{t} \frac{\Omega(t)\,\mathrm{d}t}{\epsilon^4}\right)\right],$$

where the wavenumber α and the slowly varying frequency Ω are real. The frequency and amplitude of the wave must vary on the vortex timescale in order to allow for the situation when the vortex flow is evolving in time. If the vortex flow is in equilibrium then the frequency and amplitude of the TS wave are constant.

Away from the viscous layers near $y = 0, 1$ the velocity field and pressure expand as

$$\boldsymbol{u} = \left\{\begin{pmatrix} U_0 + u_0 \\ \frac{1}{2}\epsilon^7 v_0 \\ \frac{1}{2}\epsilon^7 w_0 \end{pmatrix} + \left[\begin{pmatrix} \epsilon^6 u_1 \\ \epsilon^7 v_1 \\ \epsilon^7 w_1 \end{pmatrix} E + \text{complex conjugate}\right][1 + O(\epsilon^7)]\right\}, \tag{3.1a}$$

$$p = -\frac{12x}{Re} + \frac{p_0}{2Re^2} + [\epsilon^8 p_1 E + \text{complex conjugate}][1 + O(\epsilon^7)]. \tag{3.1b}$$

The above expansions are substituted into (2.7) and the zeroth-order system for the x-independent part of the flow satisfies the vortex equations (2.11) with $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p})$ replaced by (u_0, v_0, w_0, p_0) and with forcing terms F_1, F_2 on the right-hand sides of (2.11*c*, *d*). These forcing terms come from the interaction of the TS wave with itself and are given by

$$F_1 = 2[(-\mathrm{i}\alpha u_1 v_1^* + v_1 v_{1y}^* + w_1 v_{1z}^*) + \text{complex conjugate}], \tag{3.2a}$$

$$F_2 = 2[(-\mathrm{i}\alpha u_1 w_1^* + v_1 w_{1y}^* + w_1 w_{1z}^*) + \text{complex conjugate}]. \tag{3.2b}$$

(In these two equations * denotes complex conjugation.) The boundary conditions for (u_0, v_0, w_0, p_0) come from the no-slip condition

$$u_0 = v_0 = w_0 = 0 \quad \text{on} \quad y = 0 \text{ and } 1,$$

together with a periodicity condition in z. This follows from the fact that the forcing terms become negligible in the wall layers.

Now consider the equations obtained from the zeroth-order approximation to the TS-wave equations in the core. We find that they are identical to the corresponding equations in Bennett & Hall's linear analysis, so that writing $\bar{U} = u_0 + U_0$ we have

$$\mathrm{i}\alpha u_1 + v_{1y} + w_{1z} = 0, \quad \mathrm{i}\alpha\bar{U}u_1 + v_1\bar{U}_y + w_1\bar{U}_z = 0, \tag{3.3a, b}$$

$$\mathrm{i}\alpha\bar{U}v_1 = -p_{1y}, \quad \mathrm{i}\alpha\bar{U}w_1 = -p_{1z}, \tag{3.3c, d}$$

together with the slipping condition at the walls

$$v_1 = 0, \quad y = 0, 1.$$

Bennett & Hall simplified these equations by introducing a normalized pressure term ϕ:

$$p_1 = p_{10}(t) + (\mathrm{i}\alpha)^2 A(t)\,\phi(y,z),$$
$$u_1 = A\nabla^2\phi/2\bar{U}, \quad v_1 = -\mathrm{i}\alpha A\phi_y/\bar{U}, \quad w_1 = -\mathrm{i}\alpha A\phi_z/\bar{U},$$

where $p_{10}(t)$ and the amplitude $A(t)$ need to be determined and ϕ satisfies the core equation

$$\left.\begin{aligned} &\left(\frac{\phi_y}{\bar{U}^2}\right)_y + \left(\frac{\phi_z}{\bar{U}^2}\right)_z = 0, \quad \phi \text{ periodic in } z, \\ &\phi = 0 \quad \text{on} \quad y = 0, \quad \phi = 1 \quad \text{on} \quad y = 1. \end{aligned}\right\} \tag{3.4}$$

A related core-flow problem arises in Smith (1979). We can now eliminate u_1, v_1, and w_1 from the forcing terms in the vortex equations to give

$$F_1 = 2\alpha^2|A|^2\left\{\left[\frac{(\phi_y)^2+(\phi_z)^2}{\bar{U}^4}\right](\bar{U}^2)_y + \left[\frac{(\phi_y)^2+(\phi_z)^2}{\bar{U}^2}\right]_y\right\}, \tag{3.5a}$$

$$F_2 = 2\alpha^2|A|^2\left\{\left[\frac{(\phi_y)^2+(\phi_z)^2}{\bar{U}^4}\right](\bar{U}^2)_z + \left[\frac{(\phi_y)^2+(\phi_z)^2}{\bar{U}^2}\right]_z\right\}. \tag{3.5b}$$

We note that the last terms in these equations can, if necessary, be absorbed into the pressure derivatives in the y, z vortex equations. In the two-dimensional case $\bar{U} = \bar{U}(y), \phi = \phi(y)$ and we see that $F_2 \equiv 0$, and (using (3.5)) we note that F_1 can be completely absorbed into p_{0y}. Thus for flows that are strictly two-dimensional nonlinearity occurs only at larger values of the TS wave amplitude; see for example Hall & Smith (1982).

Because the leading-order wave equations are inviscid it is necessary to consider the flow within two viscous wall layers at $y = 0, 1$. The flow in these regions is exactly the same as in linear theory since the TS waves are sufficiently small. Thus

$$y = \epsilon^2 Y$$

in the lower viscous layer and

$$\begin{aligned} \boldsymbol{u} = \{&(\epsilon^2\lambda_0(z,t)\,Y, -\tfrac{1}{2}\epsilon^{11}\mu_{0z}(z,t)\,Y^2, \epsilon^9\mu_0(z,t)\,Y) \\ &+ [(\epsilon^6 U, \epsilon^9 V, \epsilon^7 W)\,E + \text{complex conjugate}] + \ldots\}, \end{aligned} \tag{3.6a}$$

$$p^* = \rho U_{\mathrm{m}}^2\{-12\epsilon^7 x - \epsilon^{16}\mu_{0z}\,Y + [(\epsilon^8 P + \epsilon^{10} QE) + \text{complex conjugate}] + \ldots\}, \tag{3.6b}$$

where λ_0 can be determined from $\lambda_0(z,t) = \bar{U}_y|_{y=0}$ and μ_0 from $V_{0YY}|_{y=0}$. The equations to determine U, V, W, P are

$$\left.\begin{aligned} &\mathrm{i}\alpha U + V_Y + W_z = 0, \\ &\mathrm{i}(-\Omega + \alpha\lambda_0 Y)\,U + \lambda_0 V + \lambda_{0z} YW = -\mathrm{i}\alpha P + U_{YY}, \\ &P_Y = Q_Y = P_z = 0, \quad \mathrm{i}(-\Omega + \alpha\lambda_0 Y)\,W = -Q_z + W_{YY}, \end{aligned}\right\} \tag{3.7}$$

with boundary conditions $U = V = W = 0$ on $Y = 0$, together with matching conditions with the core:

$$U \to A(t)\,\phi_{yyy}|_{y=0}/(2\lambda_0), \quad W \to 0 \quad \text{and} \quad P \to p_{10} \quad \text{as} \quad Y \to \infty,$$

and a periodicity condition in Z. An important feature of these equations is the presence of a second-order pressure term Q. We need to include this term because the z-variation of the flow is relatively fast compared to the x-variation of the waves, O(1) as opposed to $O(\epsilon)$, so the z-derivatives of this smaller pressure will come in at higher order.

These boundary-layer equations were solved by Bennett & Hall (1988) in terms of the Airy function $\mathrm{Ai}(\xi)$ to give an eigen or dispersion relation. The equations governing the wall-layer flow are then matched to the core-flow solution to give the following first-order differential equation in Q_z:

$$Q_{zz}+\frac{\lambda_{0z}}{\lambda_0}\psi Q_z=\alpha^2 p_{10}-\tfrac{1}{2}\alpha^2 A\phi_{yyy}|_{y=0}\frac{\mathrm{Ai}'(\xi_0)}{(\mathrm{i}\alpha\lambda_0)^{\frac{1}{3}}K(\xi_0)}, \tag{3.8a}$$

where
$$\psi(\xi_0)=-\left(\frac{3}{2}+\frac{\xi_0}{2\,\mathrm{Ai}(\xi_0)}[\xi_0 K(\xi_0)+\mathrm{Ai}'(\xi_0)]\right), \tag{3.8b}$$

$$\xi_0=-\frac{\Omega\,\mathrm{e}^{\mathrm{i}\pi/6}}{(\alpha\lambda_0)^{\frac{2}{3}}}\quad\text{and}\quad K(\xi_0)=\int_{\xi_0}^{\infty}\mathrm{Ai}(s)\,\mathrm{d}s. \tag{3.8c, d}$$

A similar equation involving $\lambda_1(z,t)=\bar{U}_y|_{y=1}$ arises from the boundary layer at $y=1$. By applying the periodicity condition in z, combining the two results to eliminate p_{10} and A we can determine the eigenrelation. This is somewhat unwieldy to write down but can best be calculated from the following:

$$\alpha^2=\frac{1}{2}\sum_{n=0}^{n=1}\frac{f_n(2\pi/k)}{g_n(2\pi/k)}, \tag{3.9a}$$

where $2\pi/k$ is the period of the vortices and $f_n(z)$ and $g_n(z)$ satisfy the following first-order differential equations:

$$f_{nz}+\frac{\lambda_{nz}}{\lambda_n}\psi(\xi_n)f_n=\phi_{yyy}|_{y=n}\frac{\mathrm{Ai}'(\xi_n)}{(\mathrm{i}\alpha\lambda_n)^{\frac{1}{3}}K(\xi_n)}, \tag{3.9b}$$

$$g_{nz}+\frac{\lambda_{nz}}{\lambda_n}\psi(\xi_n)g_n=1, \tag{3.9c}$$

with boundary conditions

$$f_n(0)=0,\quad g_n(0)=0, \tag{3.9d}$$

with ψ and K defined above and ξ_n given by

$$\xi_n=-\frac{\Omega\,\mathrm{e}^{\mathrm{i}\pi/6}}{(\alpha\lambda_n)^{\frac{2}{3}}},\quad n=0,1. \tag{3.9e}$$

In order to facilitate the numerical solution of the vortex equations it is convenient to eliminate the pressure from the y- and z-momentum to give

$$\left.\begin{aligned}&\{\nabla^2-\partial_t\}u_0-\tfrac{1}{2}v_0 U_0'=\mathcal{N}u_0,\\&\{\nabla^2-\partial_t\}\nabla^2 v_0+TU_0 u_{0zz}=(\mathcal{N}v_0)_{zz}-(\mathcal{N}w_0)_{zz}-\tfrac{1}{2}T(u_0^2)_{zz}\\&\qquad+2\alpha^2|A|^2\left\{\frac{(\nabla\phi)_z^2(\bar{U})_y^2-(\nabla\phi)_y^2(\bar{U})_z^2}{\bar{U}^4}\right\}_z,\\&u_0=v_0=w_0=0,\quad y=0,1,\end{aligned}\right\} \tag{3.10}$$

where $\mathcal{N}\equiv\frac{1}{2}\{v_0\,\partial_y+w_0\,\partial_z\}$ and ϕ appearing on the right-hand side of the equation for v_0 satisfies the 'core' equation (3.4).

In the absence of a Tollmien–Schlichting wave it is well known that, without any loss of generality, we can take u_0, v_0 to be even functions of z, and w_0 is then an odd function of z. In fact (3.4), (3.10) and the wall equations show that the forced equation also admits a solution with u_0, v_0 even in z with w_0 odd in z. In this case Q and the corresponding upper wall pressure together with ϕ are even functions of z. Thus at $O(1)$ wavelengths the mixed vortex–TS state is found by solving (3.10) subject to the TS wall equations (3.9*b, c*).

3.1. *The numerical method*

When the Tollmien–Schlichting waves are small, as in the linear problem, the terms F_1 and F_2 become zero to leading order and the vortex and dispersion equations then decouple. Bennett & Hall (1988) solved (3.10) with $F_1 \equiv F_2 \equiv 0$ using the method of Rogers & Beard (1969) for solving the Taylor vortex equations, by expanding the vortex velocities in Fourier sine and cosine series in z and by advancing in time to reach a steady solution. The nonlinear terms were calculated explicitly. Equation (3.4) was then solved for the wave pressure ϕ by using finite differences in both y and z and iterating to a solution. Several thousand iterations were required, the exact number depending on the method used and the step lengths in y and z. Once the skin frictions λ_0, λ_1 and pressure derivatives $\phi_{yyy}|_{y=0,1}$ had been determined α and Ω were found by solving the dispersion relation (3.9) using a fourth-order Runge–Kutta method. The Airy functions were calculated using a combination of ordinary and asymptotic series, depending on the value of the argument ξ.

We could adapt that approach to solve our unsteady nonlinear problem but we used the more efficient method which we now describe. First we fix the term $\alpha^2|A|^2$ that occurs in (3.10). Starting with a guess for $\boldsymbol{u}_0$ and ϕ we march forward in time using an Euler scheme with the u_0, v_0 equations. At each time step we update ϕ from (3.4) by performing a few iterations. The forcing terms F_1 and F_2 can then be updated for the next time step. We continue this process until $\boldsymbol{u}_0$ and ϕ have settled down to steady solutions. The dispersion relation (3.9) can then be solved for real α and Ω. Once α is known the steady amplitude $|A|$ can be determined. Even if we were to solve the pressure equation (3.4) accurately at each time step, the velocities $\boldsymbol{u}_0(t)$ that are produced before a steady solution emerges are not solutions of the physical problem. This is because we would find when we solved the eigenrelation (3.9) that in order to make Ω real we would have to have different values of α at each time step, whereas the theory requires α to be independent of time. Only when the calculation has settled down to a steady solution do we have anything that makes sense physically. This means that the algorithm may well converge on unstable solutions as well as the stable ones, although of course we will not be able to tell which is which. Because ϕ is only iterated upon a few times at each time step the overall time taken for this method is about equal to that for the linear calculation.

Since the TS-wave part of the equations is still linear the calculation of ϕ and the solution of the eigenrelation (3.9) are unchanged from Bennett & Hall. The algorithm used to step forward (3.4), (3.10) in time is virtually identical to that used by Bennett & Hall. The only difference is the presence in (3.10) of the forcing terms F_1 and F_2 which are periodic functions of z. All the periodic functions of z can be expanded in the form

$$u_0 = \sum_{-\infty}^{\infty} u_{0n}\, \mathrm{e}^{\mathrm{i}nkz}, \tag{3.11}$$

where k is the vortex wavenumber. We recall that the equations for the interaction allow a solution with u_0, v_0 and the TS wall pressure even in z and w_0, p_0, and odd in z. Thus we expect that there will be a solution of the nonlinear problem with $u_{0n} =$

u_{0-n}, etc. However, in order to allow for the possibility of solutions without this symmetry we did not force this constraint. A consequence of this is that there is a mean (independent of x, z) flow induced in the spanwise direction and an induced pressure gradient in that direction is needed to satisfy a mass flow constraint. A similar pressure function arises in the work of say DiPrima & Stuart (1975) or Hall (1984). Thus we allowed for such a function in our calculations. However, the only nonlinear solutions found had the above symmetries in z in which case there is no mean spanwise flow. It is possible that nonlinear solutions without these symmetries exist but they apparently do not bifurcate from the pure vortex flow. We postpone until §6 a discussion of the numerical results obtained following the scheme outlined above.

4. The strongly nonlinear interaction between long-wavelength Taylor vortices and Tollmien–Schlichting waves

Here it is assumed that the channel curvature parameter δ defined by (2.1) is sufficiently large to support Taylor vortices with cross-stream wavenumbers of order $\epsilon = Re^{-\frac{1}{7}}$. More precisely we consider the limits $\delta \to 0$, $Re \to \infty$ with $\tilde{T}$ defined by (2.12*b*) held fixed. In this case the basic circumferential flow can support vortices and oblique TS waves with comparable spanwise wavenumbers. The TS waves now have the structure first discussed by Smith (1979) in connection with the instability of the flow in an elliptical pipe. The overall size of the TS wave is again fixed by the condition that it should be sufficiently large in the core to drive the vortex flow at zeroth order and indeed alter the mean-flow profile. The streamwise and time dependence of the wave will again be expressed in terms of

$$E = \exp \mathrm{i}\left\{\alpha\epsilon x - \int^{t} \frac{\Omega(\tilde{t})}{\epsilon^4}\,\mathrm{d}t\right\}$$

in order to account for the possible evolution of the TS frequency as the vortex flow develops in time. The spanwise dependence of both modes is now of course entirely on the $Z = \epsilon z$ scale. In this case the appropriate expansions of the velocity and pressure fields now become

$$\boldsymbol{u} = \left\{\begin{pmatrix} U_0 + u_0 \\ \frac{1}{2}\epsilon^7 v_0 \\ \frac{1}{2}\epsilon^6 w_0 \end{pmatrix} + \ldots + \begin{pmatrix} \epsilon^5 u_1 \\ \epsilon^6 v_1 \\ \epsilon^7 w_1 \end{pmatrix} E + \text{complex conjugate} + \ldots\right\}, \tag{4.1a}$$

$$p = -12x\epsilon^7 + p_0 \tfrac{1}{2}\epsilon^{12} + \ldots + \epsilon^7 p_1 E + \text{complex conjugate} + \ldots. \tag{4.1b}$$

Here again the Taylor vortex functions u_0, v_0, w_0, p_0 and the TS functions u_1, v_1, w_1, p_1 must be matched with the appropriate expansions in the wall layers. In fact the Taylor vortex functions satisfy (2.14) but with forcing terms arising from the interaction of the Tollmien–Schlichting wave with itself. The zeroth-order core equations for the TS wave yield the solution

$$u_1 = A\bar{U}_y, \quad v_1 = -\mathrm{i}\alpha A\bar{U}, \quad p_1 = -A\alpha^2 \int_{\frac{1}{2}}^{y} \bar{U}^2\,\mathrm{d}y, \tag{4.2a, b, c}$$

$$w_1 = \frac{\alpha^2\left[A \int_{\frac{1}{2}}^{y} \bar{U}^2\,\mathrm{d}y\right]_z}{\mathrm{i}\alpha\bar{U}}. \tag{4.2d}$$

Here A is an amplitude function of $Z = \epsilon z$ and the Taylor vortex timescale t. It should be noted that A cannot vary on a faster timescale more appropriate to a TS wave alone since the Taylor vortex cannot itself respond on such a scale induced by the forcing terms. With the form of the TS wave determined in the core the forcing terms in the vortex equations in the core can be expressed in terms of A and $\bar{U}$. If the vortex pressure is eliminated we obtain

$$\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial Z} = 0, \tag{4.3a}$$

$$\{\partial_y^2-\partial_t\}\, u_0 = \tfrac{1}{2}v_0\, U_{0y}+\mathscr{M}u_0, \tag{4.3b}$$

$$\{\partial_y^2-\partial_t\}\,\partial_y^2\, v_0+\tilde{T}U_0\, u_{0ZZ} = -(\mathscr{M}w_0)_{ZZ}-\tfrac{1}{2}\tilde{T}(u_0^2)_{ZZ}+2\alpha^2\{|A|_Z^2(\bar{U}^2)_y\}_Z, \tag{4.3c}$$

$$u_0 = v_0 = w_0 = 0, \quad y = 0, 1, \tag{4.3d}$$

where $\mathscr{M} \equiv v_0\,\partial_y+w_0\,\partial_Z$.

We have anticipated above that the vortex flow in the presence of a TS wave should still satisfy the no-slip condition at the wall. In fact the spanwise momentum equation for the vortex flow in the wall layers is forced by the TS waves. However, the forcing function decays to zero like the inverse square of the distance from the wall so that the appropriate matching conditions for w_0 are that it should vanish at $y = 0, 1$.

It remains for us to determine the structure of the TS wave and Taylor vortex in the viscous wall layers. The appropriate expansions in the wall layer at $y = 0$ are

$$\boldsymbol{u} = \{[\epsilon^2\lambda_0(Z,t)\, Y, \epsilon^{11}V_0, \epsilon^8 W_0]+[(\epsilon^5 U, \epsilon^8 V, \epsilon^5 W)\, E+\text{complex conjugate}]+\dots\} \tag{4.4a}$$

$$p = -12\epsilon^7 x+\epsilon^{12}P_0+\dots+(\epsilon^7 PE+\text{complex conjugate})+\dots. \tag{4.4b}$$

The spanwise momentum equation for the vortex is now the only equation forced by the TS wave. However, the equations for the TS wave depend only on the vortex flow through the shear λ_0 so we do not solve for V_0, W_0 and P_0 here. It suffices to say that the vortex equations in the wall layer at $y = 0$ can be solved such that the solution matches with the core-flow vortex solution. The zeroth-order equations for the TS waves in the wall layer at $y = 0$ are

$$\mathrm{i}\alpha U+V_Y+W_Z = 0, \tag{4.5a}$$

$$\mathrm{i}(-\Omega+\alpha\lambda_0\, Y)\, U+\lambda_0\, V+\lambda_{0Z}\, YW = -\mathrm{i}\alpha P+U_{YY}, \tag{4.5b}$$

$$P_Y = 0, \tag{4.5c}$$

$$\mathrm{i}(-\Omega+\alpha\lambda_0\, Y)\, W = -P_Z+W_{YY}, \tag{4.5d}$$

with boundary conditions

$$U = V = W = 0, \quad Y = 0 \tag{4.5e}$$

and matching condition

$$U \rightarrow \lambda_0 A, \quad Y \rightarrow \infty. \tag{4.5f}$$

Following Smith (1979) we can solve (4.5d) to give

$$W = -\frac{P_Z\,\mathscr{L}(\xi)}{\Delta^{\frac{2}{3}}}, \tag{4.6}$$

where

$$\mathscr{L} = \mathrm{Ai}\,(\xi)\int_{\xi_0}^{\xi}\frac{\mathrm{d}s}{\mathrm{Ai}^2(s)}\int_{\infty}^{s}\mathrm{Ai}\,(s_1)\,\mathrm{d}s_1,$$

$$\xi = \Delta^{\frac{1}{3}}\left(Y-\frac{\Omega}{\alpha\lambda_0}\right), \quad \xi_0 = \frac{-\mathrm{i}^{\frac{1}{3}}\Omega}{\Delta^{\frac{2}{3}}}, \quad \Delta = \mathrm{i}\alpha\lambda_0.$$

The x-velocity component of the TS wave is then given by

$$\Delta^{\frac{5}{3}}U_\xi = -\lambda_0 P_{ZZ}\mathscr{L}'(\xi)+\tfrac{1}{3}\lambda_{0Z}P_Z\left\{3\mathscr{L}'(\xi)+\tfrac{1}{2}\mathscr{L}''''(\xi)-2\mathrm{Ai}'(\xi)\frac{\xi_0\mathscr{L}'(\xi_0)}{\mathrm{Ai}(\xi_0)}\right\}+B\,\mathrm{Ai}(\xi), \tag{4.7}$$

which matches with the core-flow solution if

$$B = \frac{\Delta^{\frac{5}{3}}\lambda_0 A}{K_0}-\tfrac{1}{2}\lambda_{0Z}P_Z\xi_0\frac{\mathscr{L}'(\xi_0)}{K_0}. \tag{4.8}$$

For convenience we shall henceforth denote any function involving Ai evaluated at ξ_0 by a 0 subscript, thus for example K_0 is K as defined by (3.8d) but with $\xi = \xi_0$. A similar expansion procedure can be carried out in the upper layer, while the core equations for the TS wave show that P and Q, the pressures in the lower and upper layers respectively, are related by

$$P-Q = \alpha^2 A\int_0^1 \bar{U}^2\,\mathrm{d}y. \tag{4.9}$$

The TS pressure P can be written using (4.5b) as

$$\mathrm{i}\alpha P = \Delta^{\frac{2}{3}}U_{\xi\xi}|_{\xi=\xi_0}$$

so that (4.7), (4.8) now give

$$P''+\frac{\lambda_0'}{\lambda_0}\psi_0 P' = \frac{\mathrm{Ai}'(\xi_0)\lambda_0^{\frac{2}{3}}[Q-P]}{K_0\alpha^2 J}+\alpha^2 P, \tag{4.10}$$

where ψ_0 is defined by (3.8b) with $\xi = \xi_0$ and

$$J = \int_0^1 \bar{U}^2\,\mathrm{d}y. \tag{4.11}$$

A similar calculation in the upper layer shows that

$$Q''+\frac{\lambda_1'}{\lambda_1}\psi_1 Q' = \alpha^2 Q+\frac{\mathrm{Ai}'(\xi_1)\lambda_1^{\frac{2}{3}}[P-Q]}{K_0\alpha^2 J}. \tag{4.12}$$

Here $\lambda_1 = -\partial\bar{U}/\partial y|_{y=1}$ and ξ_1, ψ_1 are obtained from ξ_0, ψ_0 by replacing λ_0 by λ_1 in the definition of these quantities. Thus the evolution equations (4.3), (4.9), (4.10), (4.12) determine the TS and vortex structure as the nonlinear interactions take place. The vortex velocity components u_0, v_0, w_0, the Tollmien–Schlichting amplitude A (or P, Q) and frequency Ω are all functions of the vortex timescale t. Again it should be noted that the vortex structure cannot tolerate an evolution on a faster timescale typical of TS wave growth.

The most efficient method we devised to solve the interaction equations is now described. In this method we seek a steady-state solution directly so that ∂_t is set equal to zero and values for α, $|A|_{\max}$, $\tilde{T}$ are chosen and a guess is made for $\mathscr{A}(z) = A/|A|_{\max}$. We then use a Newton iteration procedure on the steady vortex equations to find the corresponding equilibrium vortex flow. The TS wall equations are then solved for the real α and Ω which satisfy the eigenrelation. The new amplitude function $\mathscr{A}(z)$ obtained from this calculation is then used in the vortex equations to find an 'updated' vortex flow. The wall equations are then used to 'update' α, Ω and the procedure continued until the vortex solution and (α, Ω) converge for the given value of $\alpha, |A|_{max}$ and $\tilde{T}$.

It was found that the above iteration procedure converged and typically about 100 steps in $(0, 1)$ were taken to integrate the vortex equations using a fourth-order Runge–Kutta scheme. The wall equations were solved by expanding P, Q and the coefficients in (4.10), (4.12) in Fourier series in Z. It was found that about eight Fourier terms were usually adequate to solve the eigenrelation to the accuracy needed to plot the figures given later in this paper.

5. Secondary instability of two-dimensional Tollmien–Schlichting waves described by the TS–vortex interaction equations

Here we demonstrate how the interaction equations can be used to model the three-dimensional breakdown of two-dimensional TS waves in a straight channel. Suppose then that $A(Z, t)$, the wave amplitude in the core region, takes the form

$$A(Z,t) = A_0(t) + A_1 \mathrm{e}^{\mathrm{i}\omega t} \cos \beta Z, \tag{5.1}$$

where A_1 is independent of time and is small compared with the two-dimensional amplitude A_0. If $A_1 = 0$ the TS forcing term in (4.3) vanishes identically so that A_0 does not itself induce a longitudinal vortex field. This means that $A_0(t)$ evolves as the solution of a cubic amplitude equation which must be found at higher order. In fact $A_0(t)$ evolves on a longer timescale than t so that in (5.1) A_0 can be treated as a constant. We shall determine the values of A_0 at which the three-dimensional perturbation in (5.1) is neutrally stable. The precise form of the perturbation in (5.1) corresponds to two equally inclined oblique TS waves of the same amplitude.

The vortex velocity field (u_0, v_0, w_0) induced by the TS wave (5.1) can be expressed as

$$(u_0, v_0, w_0) = [4\alpha^2\beta^2 A_1 A_0 (u_{01} \sin \beta Z, v_{01} \sin \beta Z, w_{01} \cos \beta Z)\, \mathrm{e}^{\mathrm{i}\omega t} + \text{complex conjugate}] + O(A_1^2). \tag{5.2}$$

Here (u_{01}, v_{01}, w_{01}) satisfy the problem

$$\frac{\partial v_{01}}{\partial y} - \beta w_{01} = 0, \tag{5.3a}$$

$$(\partial_y^2 - \mathrm{i}\omega)\, u_{01} = \tfrac{1}{2} v_{01} U_0', \tag{5.3b}$$

$$(\partial_y^2 - \mathrm{i}\omega)\, \partial_y^2 v_{01} = -U_0 U_0', \tag{5.3c}$$

$$u_{01} = v_{01} = v_{01}' = 0, \quad y = 0, 1, \tag{5.3d}$$

which must be solved numerically (see below).

In the lower and upper wall layers P and Q expand as

$$P = P_0 + P_1 \cos \beta Z\, \mathrm{e}^{\mathrm{i}\omega t}, \quad Q = Q_0 + Q_1 \cos \beta Z\, \mathrm{e}^{\mathrm{i}\omega t},$$

and from (4.9) it follows that

$$P_1 - Q_1 = \alpha^2[\tfrac{6}{5} A_1 + 8 A_1 A_0^2 \alpha^2 \beta^2 J_1]$$

where

$$J_1 = \int_0^1 U_0 u_{01}\, \mathrm{d}y.$$

In fact it can be shown that $Q_1 = -P_1$. If the wall equations are expanded in powers of A_1, then at zeroth order we obtain

$$\alpha^2 = \frac{5.6^{\frac{2}{3}}\,\mathrm{Ai}_0'}{K_0\,\Delta^{\frac{1}{3}}}, \tag{5.4}$$

where K_0 is defined by (3.8*d*) and $\Delta = \mathrm{i}\alpha\lambda_0$ again. Equation (5.4) yields the usual neutral values of α, Ω when

$$\frac{\mathrm{Ai}_0'}{K_0} \approx 1.001\mathrm{i}^{\frac{1}{3}}, \quad \xi_0 = -2.298\mathrm{i}^{\frac{1}{3}}.$$

The condition (5.4) ensures that the two-dimensional TS wave is in neutral equilibrium. At next order we find that

$$1+\tfrac{20}{3}\beta^2\alpha^2 A_0^2 J_1 = \alpha^2\Delta^2[-\tfrac{5}{3}J_1+\tfrac{1}{2}u_{01}'(0)\,\Phi], \tag{5.5}$$

where

$$\Phi = \frac{5}{18} - \frac{\xi\,\mathrm{Ai}_0}{9}\left(\frac{\xi_0}{\mathrm{Ai}_0}+\frac{1}{K_0}\right).$$

The integral J_1 and the shear $u_{01}(0)$ depend on ω and (5.5) can be solved by taking real and imaginary parts to give

$$\beta^2 = \frac{8b}{d}, \quad \alpha^2 A_0^2 = \frac{1}{8[a-bc/a]} \tag{5.6\,a, b}$$

where

$$-\tfrac{5}{6}J_1+\tfrac{1}{2}\Phi u_{01}'(0) = a+\mathrm{i}b, \quad \tfrac{220}{3}J_1 = c+\mathrm{i}d. \tag{5.6\,c, d}$$

A numerical investigation of (5.3) shows that (5.6*a*, *b*) give $\beta^2 > 0, \alpha^2 A_0^2 > 0$ for $\omega > 59.5$. In fact the right-hand side of (5.6*b*) is positive for $\omega > 0$, while the right-hand side of (5.6*a*) $\rightarrow \infty$ when $\beta\alpha^{-1} \rightarrow \infty$, and so three-dimensional waves oriented at all angles to the flow direction can be neutral. However, the value of A_0 corresponding to β increases monotonically with $\beta\alpha^{-1}$ so that three-dimensional waves propagating almost parallel to the flow direction (i.e. with small β) are, in a sense, the most dangerous. Thus (5.6*a*, *b*) determine the amplitude of the two-dimensional TS wave which is neutrally stable to a pair of oblique waves with wavenumber β. It follows that at this amplitude there can be a secondary instability of the two-dimensional TS wave to a pair of equally inclined oblique modes.

6. Results and discussion

We first discuss our results for the interaction problem of §3 which we recall concerns Görtler vortices of wavelength comparable with the channel width. In this case finite-amplitude vortex flows are possible only for $T > T_c = 5162$ and the least stable spanwise wavenumber has $k = 3.951$. Since there are no unstable TS waves with streamwise wavenumbers $O(Re^{-\frac{1}{7}})$ and spanwise wavenumbers $O(1)$ we restricted our calculations to the case $T > T_c$ and spanwise wavenumber $k = 3.951$.

In figures 2, 3, we show the dependence of $\alpha|A|$ on Ω and α for equilibrium states corresponding to $T = 11000$, and $T = 22000$ respectively. We see that at these Taylor numbers the neutral wavenumber and frequency both decrease monotonically as $|\alpha A|$ increases. Results for $T = 5500$ have the same properties and are shown in

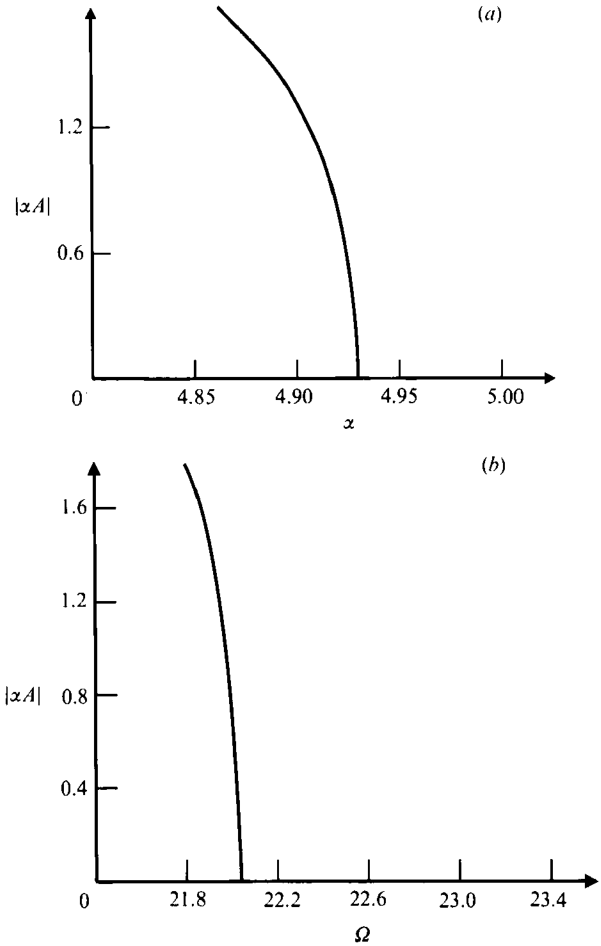

FIGURE 2. The dependence of αA on (a) α, (b) Ω for $T = 11000$, $k = 3.951$.

table 1. From these results we deduce that a fixed-frequency disturbance bifurcates supercritically with increasing α. We have not computed sufficient results to draw such a figure in detail because it would require an order-of-magnitude more computing time than that needed to generate figures 2, 3. Linear theory shows that for each of the points of figures 2, 3 the flow with infinitesimal $|A|$ is unstable. We conclude that the interaction problem at $O(1)$ vortex wavelengths leads to a supercritical bifurcation to a stable mixed TS–vortex state.

Now let us turn to our results for the interaction problem specified by (4.3), (4.9), (4.10) and (4.11) corresponding to a longer-scale spanwise dependence. In the absence of a finite-amplitude Görtler vortex we find that the neutral values of α and Ω corresponding to an oblique TS wave with $A \sim \cos z$ or $A \sim \sin z$ are

$$\alpha = 4.396, \quad \Omega = 20.456.$$

In the first instance we confine our remarks to the situation when $\tilde{T} > 23711$ so that a finite-amplitude vortex can exist in the absence of TS waves. In figure 4 we show how the neutral values of α, Ω vary in the presence of a finite-amplitude vortex at different values of the Taylor number. These values correspond to the linearized

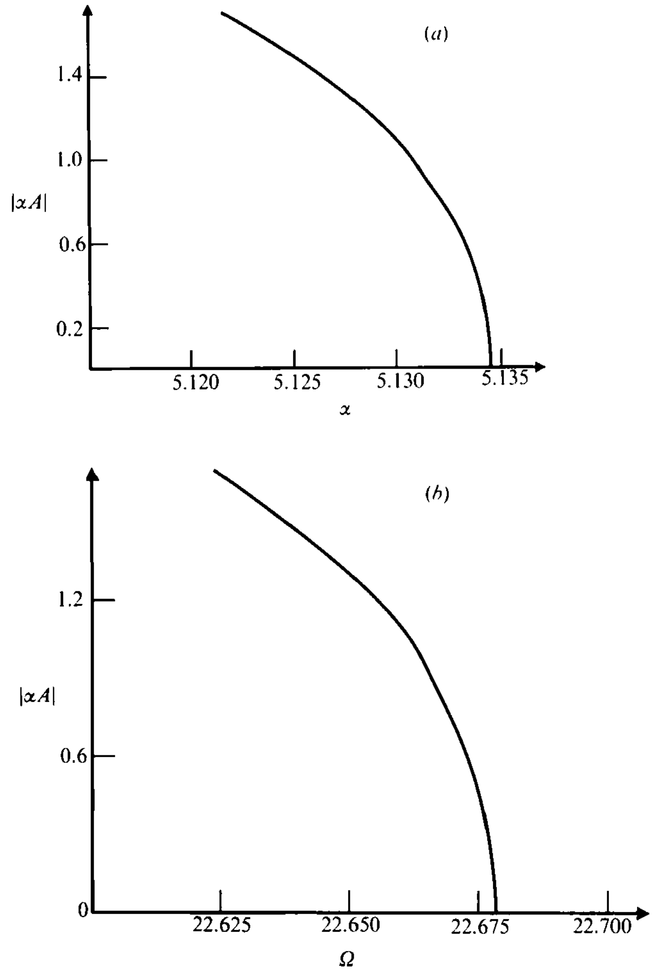

FIGURE 3. The dependence of αA on (*a*) α, (*b*) Ω for $T = 22000$, $k = 3.951$.

A	α	Ω
0.1104	4.5289	20.7614
0.1561	4.5284	20.7600
0.1913	4.5279	20.7585
0.2705	4.5266	20.7542

TABLE 1. Values of α, A and Ω for $T = 5500$ and $k = 3.951$.

problem $A \rightarrow 0$ and we see that both α and Ω increase monotonically with $\tilde{T}$. We note that a constant-frequency TS wave will change from being stable to unstable when $\tilde{T}$ is decreased through any point on the curve of figure 4 for $\tilde{T}$ greater than its linear neutral value ~ 23711.

Next suppose that the Taylor number is held fixed and the neutral values of α and Ω are calculated for different values of $|A|_{max}$. When $|A|_{max} \rightarrow 0$ then the neutral values of α and Ω must tend back to the neutral values appropriate to the current value of the Taylor number. The results of such a calculation are shown in figure 5

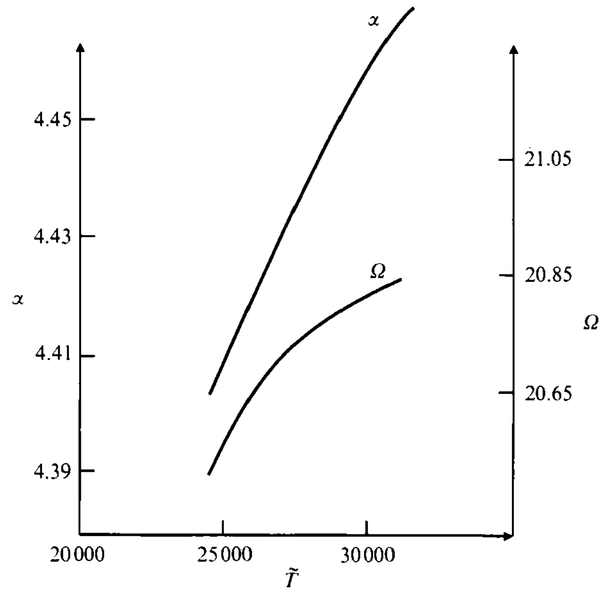

FIGURE 4. The dependence of α and Ω on $\tilde{T}$.

at two typical values of the Taylor number. We see that α and Ω increase monotonically when $|A|_{max}$ increases. We conclude that the bifurcation picture for any constant-frequency TS wave will be as sketched in figure 6. At first sight it appears that this is a subcritical bifurcation to a mixed vortex–TS state. However, we found earlier that a constant-frequency linear TS wave changes from being stable to unstable when $\tilde{T}$ is decreased, so that, if we associate the terms 'subcritical' and 'supercritical' with the linearly stable and unstable regions, we conclude that then a constant-frequency disturbance undergoes a supercritical bifurcation to a finite-amplitude state. A weakly nonlinear analysis of (4.3), (4.9), (4.10), (4.11) shows that when this bifurcation occurs the finite-amplitude solution is stable and we do not expect that there will be any finite time singularities of the full time-dependent problem associated with the finite-amplitude state.

Now let us consider the situation when $\tilde{T} < 23711$ so that in the absence of a Tollmien–Schlichting wave there is no vortex activity. In order to see the differences which emerge in this case we shall now indicate briefly how an amplitude equation typical of those obtained for weakly nonlinear stability problems using the Stuart–Watson method can be retrieved. Suppose then that the Taylor number $\tilde{T}$ is held fixed and that the neutral values of α and Ω are given by

$$\alpha = \alpha_N, \quad \Omega = \Omega_N. \tag{6.1}$$

These are the neutral values appropriate to $|A|_{max} \to 0$, so that if we now write

$$|A|_{max} = \delta^{\frac{1}{2}},$$

where δ is small and positive we anticipate that with $\tilde{T}$ held fixed the appropriate expansions of α and Ω become

$$\alpha = \alpha_N + \delta\tilde{\alpha} + \ldots, \quad \Omega = \Omega_N + \delta\tilde{\Omega} + \ldots. \tag{6.2a, b}$$

We see from (4.3) that u_0, v_0, w_0 in (4.3) must be expanded as

$$u_0 = \delta\tilde{u}\cos 2z + \ldots, \quad v_0 = \delta\tilde{v}\cos 2z + \ldots, \quad w_0 = \delta\tilde{w}\sin 2z + \ldots,$$

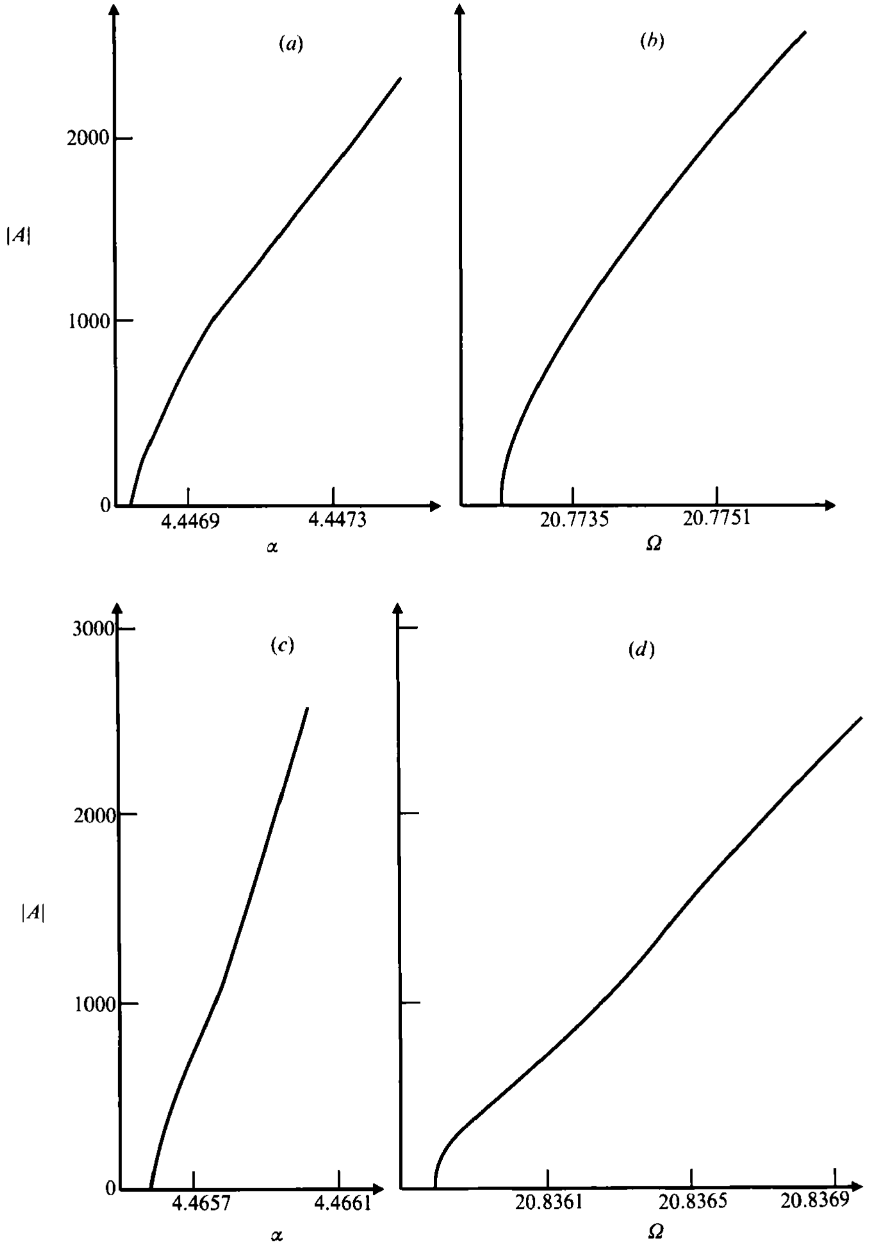

FIGURE 5. (*a, b*) The dependence of A_{max} on (*a*) α, (*b*) Ω for the case $\tilde{T} = 28511$. (*c, d*) The dependence of A_{max} on (*c*) α, (*d*) Ω for the case $\tilde{T} = 31000$.

where $\tilde{u}, \tilde{v}, \tilde{w}$ satisfy

$$\frac{d\tilde{v}}{dy} + 2\tilde{w} = 0, \quad \frac{d^2\tilde{u}}{dy^2} = \tfrac{1}{2}\tilde{v}U_{0y}, \quad \frac{d^4\tilde{v}}{dy^4} - 4\tilde{T}U_0 = -4\alpha^2\tilde{T}U_{0y}^2, \qquad (6.3a, b)$$

$$\tilde{u} = \tilde{v} = \tilde{w} = 0, \quad y = 0, 1. \qquad (6.3d)$$

We note that the homogeneous form of (6.3) has a solution if $4\tilde{T} \approx 23711$, so that $\tilde{u}, \tilde{v}, \tilde{w}$ then become singular near $\tilde{T} = \tilde{T}^+ \approx 5978$. In fact these functions behave like $1/(\tilde{T} - \tilde{T}^+)$ so that $\hat{u}, \hat{v}, \hat{w}$ necessarily change sign at $\tilde{T}^+$. The wall equations (4.10) and

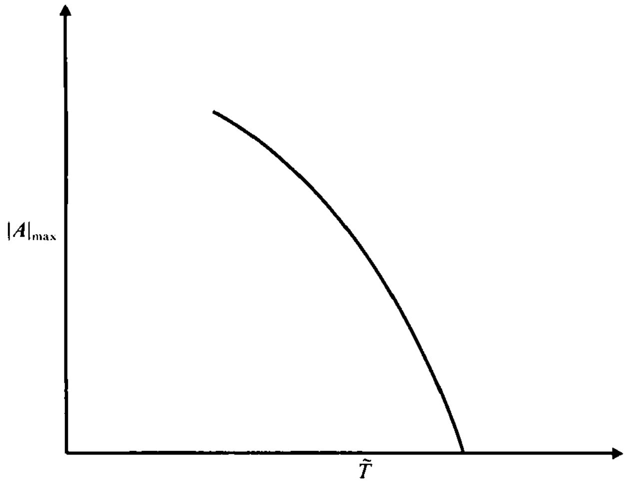

FIGURE 6. A sketch of the bifurcation diagram for a constant-frequency TS wave.

(4.12) together with (4.11) are then found to have a solution only if $\tilde{\alpha}, \tilde{\Omega}$ satisfy an equation of the form

$$\tilde{\alpha}[c+\mathrm{i}d]+\tilde{\Omega}[e+\mathrm{i}f]+g+\mathrm{i}h=0. \tag{6.4}$$

Here c, d, e, f, g and h are real constants, the first four of which arise from the expansions of α and Ω in (4.10)–(4.12), whilst g and h arise from $\hat{u}$, the order-δ correction to the core flow. By taking real and imaginary parts in (6.4) we can solve for $\tilde{\Omega}$ and $\tilde{\alpha}$. Our numerical solution of the full problem for $\tilde{T} > 23711$ suggests that there Ω is always positive. However, because of the singularity of $\tilde{u}$ at $\tilde{T}^{+}$ it follows that $\tilde{\Omega}$ must change sign at $\tilde{T} = \tilde{T}^{+}$.

It follows from the above that in the neighbourhood of $\tilde{T}^{+}$ there is a change in the bifurcation structure of the mixed vortex–TS state; thus on one side of this Taylor number the bifurcation will be supercritical whilst on the other side it will be subcritical. In fact this structure is discussed by Hall & Smith (1988). In the present investigation the iteration scheme used to solve the interaction problem of §4 failed to converge for Taylor numbers less than 25000. Until the method failed to converge it was found that the bifurcation to the mixed state was always supercritical. This suggests that at sufficiently high Taylor numbers the time-dependent interaction problem does not have a finite time singularity but that at lower Taylor numbers there could be such a singularity associated with the subcritical bifurcation.

In conclusion then we note that we have been able to find strongly nonlinear disturbances to curved channel flows. The size of the disturbances we have considered is such that they cannot be described by weakly nonlinear calculations based on the Stuart–Watson method. The only alternative description of such large disturbances would have to be based on the full Navier–Stokes equations. However, the advantage of the approach we have derived is that the TS-wave dependence of the flow is accounted for analytically. Thus the scale of the computations to be carried out using our approach is significantly smaller than would be the case with the full Navier–Stokes equations. Moreover the results obtained here and in the related boundary layer work of Hall & Smith (1989*b*) encourage us to believe that the vortex–wave interaction approach is capable of capturing many of the significant stages of some routes of transition.

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